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On conservation laws in fourth-order potential barriers

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Abstract. Wave propagation across potential barriers is an important aspect of theoretical physics. In this paper, the waves are described by fourth-order ordinary differential equations, giving two modes in which wave-energy can be carried in the non-dissipative medium. A conserved quantity controlling the relative distribution of energy into the different channels of propagation is derived, using only the properties of the governing equations.

1. Introduction

In many wave propagation problems, potential barriers play a major role in determining the response of a medium to small perturbations about a stable equilibrium. In most classical examples, the model equations are second order ordinary differential equations, so that the barrier is concerned only with transmission and reflection of the single mode supported by such a description. (Throughout we assume that the medium is non-dissipative.)

A plasma, however, can support many types of wave simultaneously, so that any barrier problem in this context properly should account for the many possible ways in which energy can be transported by oscillations in the medium.

In particular, a cold plasma (which is one in which the hydrodynamic pressure is not important) can support two kinds of wave: ordinary and extraordinary (Stix 1962). Under certain circumstances, the equilibrium can be arranged such that a localised barrier is presented to any wavemotion in the plasma (Diver and Laing 1989). This then is the motivation for studying transmission, reflection and mode conversion in such fourth-order systems, and for deriving a conserved quantity governing the forward and backward scattering of energy into the available channels.

2. Model equations

A typical model equation describing a physical system capable of supporting two distinct types of wave is the following ordinary differential equation (ODE):

$$y^{(iv)} + \alpha y^{(ii)} + i\beta y' + \gamma y = 0 \tag{1}$$

where we take the parameters α, β and γ to be real constants. Note that the coefficient of the odd order derivative term is purely imaginary, since then the polynomial in k arising from Fourier transformation has real coefficients. That is, on substituting $y = e^{ikz}$ into (1), k must be a solution of

$$k^4 - \alpha k^2 - \beta k + \gamma = 0. \tag{2}$$

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3. Mathematical technique

Before describing the actual mathematical method used to calculate the conservation law, it is useful to define the Wronskian $W(u, v)$, and to detail certain of its properties which will prove useful in the full analysis.

The Wronskian of two dependent variables u, v is defined by

$$W(u, v) = uv' - u'v \tag{3}$$

where ′ denotes d/dz. Differentiation of $W(u, v)$ yields the following relations:

$$W'(u, v) = uv^{(ii)} - u^{(ii)}v \qquad W^{(ii)}(u, v) = W(u', v') + uv^{(iii)} - u^{(iii)}v$$
$$W^{(iii)}(u, v) = 2W'(u', v') + uv^{(iv)} - u^{(iv)}v. \tag{4}$$

The complex conjugate of equation (1) is

$$y^{*(iv)} + \alpha y^{*(ii)} - \mathrm{i}\beta y^{*\prime} + \gamma y^* = 0. \tag{5}$$

Using these two equations (1) and (5), it is possible to eliminate each derivative term in turn by cross multiplication, in such a way that the relations (4) may be used to simplify the remainders. Thus multiplying (1) by $y^{*(iv)}$, (5) by $y^{(iv)}$ and subtracting yields

$$\alpha(y^{*(iv)}y^{(ii)} - y^{*(ii)}y^{(iv)}) + \mathrm{i}\beta(y^{*(iv)}y' + y^{*\prime}y^{(iv)}) + \gamma(y^{*(iv)}y - y^*y^{(iv)}) = 0. \tag{6}$$

In a similar manner, we can construct

$$y^{*(ii)}y^{(iv)} - y^{*(iv)}y^{(ii)} + \mathrm{i}\beta(y^{*(ii)}y' + y^{*\prime}y^{(ii)}) + \gamma(y^{*(ii)}y - y^*y^{(ii)}) = 0 \tag{7}$$

$$y^{(iv)}y^{*\prime} + y^{*(iv)}y' + \alpha(y^{(ii)}y^{*\prime} + y^{*(ii)}y') + \gamma(yy^{*\prime} + y^*y') = 0 \tag{8}$$

and finally,

$$y^{(iv)}y^* - y^{*(iv)}y + \alpha(y^{(ii)}y^* - y^{*(ii)}y) + \mathrm{i}\beta(y'y^* + y^{*\prime}y) = 0. \tag{9}$$

Defining the function $\psi(u, v)$ by

$$\psi(u, v) = u'v^{(iii)} + u^{(iii)}v' - u^{(ii)}v^{(ii)} \tag{10}$$

and noting that

$$\psi'(u, v) = u'v^{(iv)} + u^{(iv)}v'$$

we can now write the first integrals of equations (6)-(9) as

$$\alpha W(y'', y^{*\prime\prime}) + \mathrm{i}\beta\psi + \gamma[W''(y, y^*) - 2W(y', y^{*\prime})] = \chi_1 \tag{11}$$

$$W(y^{*\prime\prime}, y'') + \mathrm{i}\beta[y^{*\prime}y'] - \gamma W(y^*, y) = \chi_2 \tag{12}$$

$$\psi(y^*, y) + \alpha y^{*\prime}y' + \gamma y^*y = \chi_3 \tag{13}$$

$$W''(y^*, y) - 2W(y^{*\prime}, y') + \alpha W(y^*, y) + \mathrm{i}\beta y^*y = \chi_4 \tag{14}$$

where the χ_i are constants.

Since there are only three parameters in the problem (α, β and γ), we expect only three of the above equations to be independent. To show that this is indeed the case, consider the first of the set:

$$\begin{aligned}\chi_1 &= \alpha W(y^{*\prime\prime}, y'') - \mathrm{i}\beta\psi(y^*, y) + \gamma[W''(y^*, y) - 2W(y^{*\prime}, y')] \\ &= \alpha W(y^{*\prime\prime}, y'') - \mathrm{i}\beta\psi(y^*, y) + \gamma[\chi_4 - \mathrm{i}\beta y^*y - \alpha W(y^*, y)] \qquad \text{by (14)} \\ &= \alpha[\chi_2 - \mathrm{i}\beta y^{*\prime}y'] - \mathrm{i}\beta\psi(y^*, y) + \gamma[\chi_4 - \mathrm{i}\beta y^*y] \qquad \text{by (12)} \\ &= -\mathrm{i}\beta[\alpha y^{*\prime}y' + \psi(y^*, y) + \gamma y^*y] + \alpha\chi_2 + \gamma\chi_4\end{aligned}$$

which is essentially equation (13). Henceforth, the complete set of conserved quantities can be taken as

$$W(y^{*''}, y'') + i\beta[y^{*'}y'] - \gamma W(y^*, y) = N \tag{15}$$

$$W''(y^*, y) - 2W(y^{*'}, y') + \alpha W(y^*, y) + i\beta y^* y = P \tag{16}$$

$$\psi(y^*, y) + \alpha y^{*'} y' + \gamma y^* y = Q. \tag{17}$$

These three quantities are invariants for the solution of (1).

4. Physical model

The motivation for deriving (15)-(17) came from considering a plasma, capable of supporting two distinct modes simultaneously, which possessed a finite region in which the modes had a different wavelength from the rest of the solution space (Diver and Laing 1989).

This problem may be viewed as an extension to the classical second-order wave barrier problems of quantum mechanics, which are concerned only with one particular wavemode (see for example, Heading 1962).

We are concerned, in this paper, with a genuine four channel scattering problem. Thus, consider the situation in which the real line is divided into three regions, labelled I, II and III. Region II is finite, and is bounded on either side by the semi-infinite domains I and III. The wave characteristics in I and III are taken to be identical, but different from those in II. In all regions, waves are described by an ODE similar to (1), with the allowable modes defined by the roots of polynomials similar to (2).

In every region, each of the invariants (15)-(17) must hold. Consequently, we may apply each formula, yielding for example,

$$W(y_I^{*''}, y_I'') + i\beta_I[y_I^{*'}y_I'] - \gamma_I W(y_I^*, y_I) = N_I$$

$$W''(y_I^*, y_I) - 2W(y_I^{*'}, y_I') + \alpha_I W(y_I^*, y_I) + i\beta_I y_I^* y_I = P_I$$

$$\psi(y_I^*, y_I) + \alpha_I y_I^{*'} y_I' + \gamma_I y_I^* y_I = Q_I$$

in region I, taking the parameters to have the values α_I, β_I and γ_I. This can be repeated in both II and III, with the obvious extension of notation.

Note that each invariant is unchanged in a region, but need not be identical in value in distinct regions, since the parameters are different. Although the parameters in regions I and III are the same, the intervening region II then prevents the direct matching of the invariants between the two semi-infinite solution spaces. This is because the differential equations are not defined at the interfaces, since the coefficients α, β and γ are ambiguous there.

Contrast this with the situation in the second-order barrier, in which the invariant is the Wronskian, and is independent of any physical parameter (Heading 1962).

However, in order to define a fourth-order equation in each region, we must have continuity in y and its first three derivatives. We will preserve this feature of the solution across the interface, and in this way, match the form of solution in each region using the invariants. This will generate a formula connecting the incident waves with the transmitted, analogous to the reflection formula of the classical barrier.

Defining the notation

$$[q] = q_I - q_{II}$$

for any parameter q, and further denoting the value of y at the interface between regions I and II by $\hat{y}$, and between II and III by $\tilde{y}$, we can construct the differences between invariants at each interface:

$$[\mathrm{i}\beta]\hat{y}^{*\prime}\hat{y}' - [\gamma]W(\hat{y}^*, \hat{y}) = N_{\mathrm{I}} - N_{\mathrm{II}} \tag{18}$$

$$[\alpha]W(\hat{y}^*, \hat{y}) + [\mathrm{i}\beta]\hat{y}\hat{y}^* = P_{\mathrm{I}} - P_{\mathrm{II}} \tag{19}$$

$$[\alpha]\hat{y}'\hat{y}^{*\prime} + [\gamma]\hat{y}\hat{y}^* = Q_{\mathrm{I}} - Q_{\mathrm{II}} \tag{20}$$

and

$$+\langle \mathrm{i}\beta\rangle\tilde{y}^{*\prime}\tilde{y}' - \langle\gamma\rangle W(\tilde{y}^*, \tilde{y}) = N_{\mathrm{II}} - N_{\mathrm{III}} \tag{21}$$

$$+\langle\alpha\rangle W(\tilde{y}^*, \tilde{y}) + \langle \mathrm{i}\beta\rangle\tilde{y}\tilde{y}^* = P_{\mathrm{II}} - P_{\mathrm{III}} \tag{22}$$

$$+\langle\alpha\rangle\tilde{y}'\tilde{y}^{*\prime} + \langle\gamma\rangle\tilde{y}\tilde{y}^* = Q_{\mathrm{II}} - Q_{\mathrm{III}} \tag{23}$$

where $\langle q\rangle$ denotes $q_{\mathrm{II}} - q_{\mathrm{III}}$ for parameter q. Note that the terms independent of the parameters must vanish, since y and its derivatives are continuous across an interface. Eliminating the terms involving $\hat{y}$ from equations (18)-(20), we have

$$[\alpha](N_{\mathrm{I}} - N_{\mathrm{II}}) - [\mathrm{i}\beta](Q_{\mathrm{I}} - Q_{\mathrm{II}}) + [\gamma](P_{\mathrm{I}} - P_{\mathrm{II}}) = 0 \tag{24}$$

defining a quantity which is conserved in passing from region I to region II.

In the same way, equations (21)-(23) yield

$$\langle\alpha\rangle(N_{\mathrm{II}} - N_{\mathrm{III}}) - \langle \mathrm{i}\beta\rangle(Q_{\mathrm{II}} - Q_{\mathrm{III}}) + \langle\gamma\rangle(P_{\mathrm{II}} - P_{\mathrm{III}}) = 0. \tag{25}$$

In our particular example, $\langle q\rangle = -[q]$, since region III has identical characteristics to region I. However, (24) and (25) are quite general conservation laws, prescribing quantities which are unvarying on crossing into neighbouring regions.

Returning to our particular case, we can summarise (24) and (25) in the following law:

$$\begin{gathered}[\alpha]\delta N - [\mathrm{i}\beta]\delta Q + [\gamma]\delta P = 0 \\ \delta N = N_{\mathrm{I}} - N_{\mathrm{III}} \qquad \delta P = P_{\mathrm{I}} - P_{\mathrm{III}} \qquad \delta Q = Q_{\mathrm{I}} - Q_{\mathrm{III}}.\end{gathered} \tag{26}$$

This is then our connection formula, allowing information about the solution in region I to be communicated to region III.

5. Explicit example

In order to reveal the physical implications of (27), consider the following example, in which we have chosen $\beta = 0$, in order to simplify the interpretation of the solution forms. The polynomial form (2) is then biquadratic, giving two modes propagating both backwards and forwards in space.

Since in each of the three regions of solution space, the ODE has real and constant coefficients, the solution can be written down as

in region I ($-\infty < z < -L$): $$y_{\mathrm{I}} = \mathrm{e}^{\mathrm{i}k_1 z} + C_1\,\mathrm{e}^{-\mathrm{i}k_1 z} + C_2\,\mathrm{e}^{-\mathrm{i}k_2 z} \tag{27}$$

in region II ($-L < z < L$): $$y_{\mathrm{II}} = D_1\,\mathrm{e}^{\mathrm{i}l_1 z} + D_2\,\mathrm{e}^{-\mathrm{i}l_1 z} + D_3\,\mathrm{e}^{\mathrm{i}l_2 z} + D_4\,\mathrm{e}^{-\mathrm{i}l_2 z} \tag{28}$$

in region III ($L < z < \infty$): $$y_{\mathrm{III}} = F_1\,\mathrm{e}^{\mathrm{i}k_1 z} + F_2\,\mathrm{e}^{\mathrm{i}k_2 z}. \tag{29}$$

This form of solution describes a wave of unit amplitude and of wavenumber k_1 incident from $-\infty$ and encountering a 'barrier' at $z = -L$, leading to reflected waves with amplitudes C_1 of the incident wavenumber k_1, and C_2 of the companion wavenumber k_2. The transmission side, i.e. region III, has waves travelling in the positive z direction only, with amplitudes F_1 and F_2, and the same wavenumbers as region I. The finite barrier, region II, is the segment in which the wavenumbers are different constants from those elsewhere on the real line.

The physical implications of the connection formula are revealed when the solutions y_{I} and y_{III} are substituted from (27) and (29) into (26). Note that the parameters α and γ are readily identified, for example

$$\alpha_{\mathrm{I}} = k_1^2 + k_2^2 \qquad \gamma_{\mathrm{I}} = k_1^2 k_2^2$$

in regions I and III. The task is then to evaluate the invariants N and P in each region, using (27) and (29) as the appropriate form for y in each case.

The algebra is tedious, and lengthy, but is simplified by recognising that no cross terms play a role in the formulae (since they are position dependent). Thus we have

$$N_{\mathrm{I}} = (k_1^2 - k_2^2)(k_1^3(1 - |C_1|^2) + k_2^3|C_2|^2) \tag{30}$$

$$N_{\mathrm{III}} = (k_1^2 - k_2^2)(k_1^3|F_1|^2 - k_2^3|F_2|^2) \tag{31}$$

$$P_{\mathrm{I}} = -(k_1^2 - k_2^2)(k_1(1 - |C_1|^2) + k_2|C_2|^2) \tag{32}$$

$$P_{\mathrm{III}} = -(k_1^2 - k_2^2)(k_1|F_1|^2 - k_2|F_2|^2). \tag{33}$$

Finally, combining these expressions, we arrive at

$$\begin{aligned} 1 &= |C_1|^2 + |F_1|^2 - \frac{k_2}{k_1}\rho(|C_2|^2 + |F_2|^2) \\ \rho &= \frac{k_2^2[\alpha] - [\gamma]}{k_1^2[\alpha] - [\gamma]} = \frac{(k_2^2 - l_1^2)(k_2^2 - l_2^2)}{(k_1^2 - l_1^2)(k_1^2 - l_2^2)}. \end{aligned} \tag{34}$$

This relation governs the distribution of wave-energy across the available propagation channels.

6. Discussion

Equation (34) is the conservation law for our model system. The formula depends on the propagation characteristics of the connecting region, in such a way that if any one of the two central wavenumbers is equal to the incident wavenumber, then the second wavesolution is not excited in region III by passing through region II. In this case we say that there is no *mode conversion*. This agrees with physical intuition, in that waves propagate throughout with the same wavelength, and so the barrier is transparent to the incident wave.

It is unusual for a conservation law to depend on the precise mechanics of the interaction region. However, in most cases, this is because conservation laws are presented for second-order systems, in which there is only one possible wavemode, and so the system parameters are absent from the definition of the invariant. In this higher order model, distinct modes have different intrinsic wave-energies, and this is reflected in the explicit role of the wavenumbers in the invariants. The matching process across each interface must then take account of the relative wavenumbers available for the propagation of energy. Since the governing ODEs do not apply at the interface, the minimum possible assumption about the solution is its continuity, and that of its derivatives, despite the ambiguity of the parameters. This is an extension of the classical barrier, but to a more richly structured problem.

The conservation law (34) has been verified independently, by numerical simulation. The results are detailed in our earlier paper (Diver and Laing 1989) and need not be repeated here. Suffice it to say that the result is a physically meaningful one, and has far reaching implications for the treatment of higher-order wave problems.

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